

The dynamical problem for a non self-adjoint Hamiltonian

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Abstract. We review some recent results of the so-called quasi-hermitian quantum mechanics, with particular focus on the quantum dynamics both in the Schrödinger and in the Heisenberg representations. The role of Krein spaces is also discussed.

Mathematics Subject Classification (2000). Primary 47B50; Secondary 81Q65 47N50 81Q12 47B36 46C20.

Keywords. metrics in Hilbert spaces, hermitizations of a Hamiltonian.

1. Introduction

In the analysis of the dynamics of a closed quantum system \mathcal{S} a special role is played by the energy H , which is typically the self-adjoint operator defined by the sum of the kinetic energy of \mathcal{S} and of the potential energy giving rise to the conservative forces acting on \mathcal{S} . The *most common* approaches in the description of \mathcal{S} are the following:

1. *the algebraic description (AD)*: in this approach the observables of \mathcal{S} are elements of a C^* -algebra \mathfrak{A} (which coincides with $B(\mathcal{H})$ for some Hilbert space \mathcal{H}). This means, first of all, that \mathfrak{A} is a vector space over \mathbb{C} with a multiplication law such that $\forall A, B \in \mathfrak{A}$, $AB \in \mathfrak{A}$. Also, two such elements can be summed up and the following properties hold: $\forall A, B, C \in \mathfrak{A}$ and $\forall \alpha, \beta \in \mathbb{C}$ we have

$$A(BC) = (AB)C, \quad A(B + C) = AB + AC, \quad (\alpha A)(\beta B) = \alpha\beta(AB).$$

An involution is a map $*$: $\mathfrak{A} \rightarrow \mathfrak{A}$ such that

$$A^{**} = A, \quad (AB)^* = B^*A^*, \quad (\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$$

A $*$ -algebra \mathfrak{A} is an algebra with an involution $*$. \mathfrak{A} is a *normed algebra* if there exists a map, *the norm of the algebra*, $\|\cdot\| : \mathfrak{A} \rightarrow \mathbb{R}_+$, such that:

$$\begin{aligned} \|A\| &\geq 0, \quad \|A\| = 0 \iff A = 0, \quad \|\alpha A\| = |\alpha| \|A\|, \\ \|A + B\| &\leq \|A\| + \|B\|, \quad \|AB\| \leq \|A\| \|B\|. \end{aligned}$$

If \mathfrak{A} is complete wrt $\|\cdot\|$, then it is called a *Banach algebra*, or a *Banach $*$ -algebra* if $\|A^*\| = \|A\|$. If further $\|A^*A\| = \|A\|^2$ holds for all $A \in \mathfrak{A}$, then \mathfrak{A} is a *C^* -algebra*.

The *states* are linear, positive and normalized functionals on \mathfrak{A} , which look like $\rho(\hat{A}) = \text{tr}(\hat{\rho}A)$, where $\mathfrak{A} = B(\mathcal{H})$, $\hat{\rho}$ is a trace-class operator and tr is the trace on \mathcal{H} . This means in particular that

$$\rho(\alpha_1 A + \alpha_2 B) = \alpha_1 \rho(A) + \alpha_2 \rho(B)$$

and that, if \mathfrak{A} has the identity $\mathbb{1}$,

$$\rho(A^*A) \geq 0; \quad \rho(\mathbb{1}) = 1.$$

An immediate consequence of these assumptions, and in particular of the positivity of ρ , is that ρ is also continuous, i.e. that $|\rho(A)| \leq \|A\|$ for all $A \in \mathfrak{A}$.

The dynamics in the Heisenberg representation for the closed quantum system \mathcal{S} is given by the map

$$\mathfrak{A} \ni A \rightarrow \alpha^t(A) = U_t A U_t^\dagger \in \mathfrak{A}, \quad \forall t$$

which defines a 1-parameter group of $*$ -automorphisms of \mathfrak{A} satisfying the following conditions

$$\begin{aligned} \alpha^t(\lambda A) &= \lambda \alpha^t(A), \quad \alpha^t(A + B) = \alpha^t(A) + \alpha^t(B), \\ \alpha^t(AB) &= \alpha^t(A) \alpha^t(B), \quad \|\alpha^t(A)\| = \|A\|, \quad \text{and} \quad \alpha^{t+s} = \alpha^t \alpha^s. \end{aligned}$$

In the Schrödinger representation the time evolution is the dual of the one above, i.e. it is the map between states defined by $\hat{\rho} \rightarrow \hat{\rho}_t = \alpha^{t*} \hat{\rho}$.

2. *the Hilbert space description (HSD)*: this is much simpler, at a first sight. We work in some fixed Hilbert space \mathcal{H} , somehow related to the system we are willing to describe, and we proceed as follows:

each observable A of the physical system corresponds to a self-adjoint operator \hat{A} in \mathcal{H} ;

the pure states of the physical system corresponds to normalized vectors of \mathcal{H} ;

the expectation values of A correspond to the following mean values: $\langle \psi, \hat{A} \psi \rangle = \rho_\psi(\hat{A}) = \text{tr}(P_\psi \hat{A})$, where we have also introduced a projector operator P_ψ on ψ and tr is the trace on \mathcal{H} ;

the states which are not pure, i.e. the mixed states, correspond to convex linear combinations $\hat{\rho} = \sum_j w_j \rho_{\psi_j}$, with $\sum_j w_j = 1$ and $w_j \geq 0$ for all j ;

the dynamics (in the Schrödinger representation) is given by a unitary operator $U_t := e^{iHt/\hbar}$, where H is the self-adjoint energy operator, as follows:

$\hat{\rho} \rightarrow \hat{\rho}_t = U_t^\dagger \hat{\rho} U_t$. In the Heisenberg representation the states do not evolve in time while the operators do, following the *dual* rule: $\hat{A} \rightarrow \hat{A}_t = U_t \hat{A} U_t^\dagger$, and the Heisenberg equation of motion is satisfied: $\frac{d}{dt} \hat{A}_t = \frac{i}{\hbar} [H, \hat{A}_t]$. It is very well known that these two different representations have the same physical content: indeed we have $\hat{\rho}(\hat{A}_t) = \hat{\rho}_t(\hat{A})$, which means that what we measure in experiments, that is the time evolution of the mean values of the observables of \mathcal{S} , do not depend on the representation chosen.

The *AD* is especially useful when \mathcal{S} has an infinite number of degrees of freedom, [1, 2, 3], while the *HSD* is quite common for ordinary quantum mechanical systems, i.e. for those systems with a finite number of degrees of freedom. The reason why the algebraic approach to ordinary quantum mechanics is not very much used in this simpler case follows from the following von Neumann uniqueness theorem: *for finite quantum mechanical systems there exists only one irreducible representation*. This result is false for systems with infinite degrees of freedom (briefly, in QM_∞), for which *AD* proved to be useful, for instance in the description of phase-transitions [4].

As we have already said, in the most common applications of quantum theory the self-adjoint Hamiltonian is just the sum of a kinetic plus an interaction term, and the Hilbert space in which the model is described is usually $\mathcal{H} := \mathcal{L}^2(\mathbb{R}^D)$, for $D = 1, 2$ or 3 . Of course any unitary map defined from \mathcal{H} to any (in general different) other Hilbert space $\tilde{\mathcal{H}}$ does not change the physics which is contained in the model, but only provides a different way to extract the results from the model itself. In particular, the mean values of the different observables do not depend from the Hilbert space chosen, as far as the different representations are connected by unitary maps. These observables are other self-adjoint operators whose eigenvalues are interesting for us since they have some physical meaning. For a quantum particle moving in D -dimensional Euclidian space, for example, people usually work with the position operator \mathbf{q} , whose eigenvalues are used to label the wave function $\psi(\vec{x}, t)$ of the system, which is clearly an element of $\mathcal{L}^2(\mathbb{R}^D)$. It might be worth reminding that we are talking here of *representations* from two different points of view: the Heisenberg and the Schrödinger representations are two (physical) equivalent ways to describe the dynamics of \mathcal{S} , while in the *AD* a representation is a map from \mathfrak{A} to $B(\mathcal{H})$ for a chosen \mathcal{H} which preserve the algebraic structure of \mathfrak{A} . Not all these kind of representations are unitarily equivalent, and for this reason they can describe different physics (e.g. different phases of \mathcal{S}), [1, 2, 3].

Recently, Bender and Boettcher [5] emphasized that many Hamiltonians H which look unphysical in \mathcal{H} may still be correct and physical, provided only that the conservative textbook paradigm is replaced by a modification called **PT**-symmetric quantum mechanics (PTSQM, cf., e.g., reviews [6, 7, 8, 9] for more details). Within the PTSQM formalism the operators **P** and **T** (which characterize a symmetry of the quantum system in question) are usually pre-selected as parity and time reversal, respectively.

2. Quantization recipes using non-unitary Dyson mappings Ω

The main appeal of the PTSQM formalism lies in the permission of selecting, for phenomenological purposes, various new and nonstandard Hamiltonians exhibiting the manifest non-hermiticity property $H \neq H^\dagger$ in \mathcal{H} . It is clear that these operators cannot generate unitary time evolution in \mathcal{H} via exponentiation, [10], but this does not exclude [11] the possibility of finding an unitary time evolution in a different Hilbert space, not necessarily uniquely determined [12], which, following the notation in [9], we indicate as $\mathcal{H}^{(P)}$, P standing for *physical*. Of course, the descriptions of the dynamics in \mathcal{H} and $\mathcal{H}^{(P)}$ cannot be connected by a unitary operator, but still other possibilities are allowed and, indeed, these different choices are those relevant for us here.

In one of the oldest applications of certain specific Hamiltonians with the property $\hat{H} \neq \hat{H}^\dagger$ in \mathcal{H} in the so called interacting boson models of nuclei [11] it has been emphasized that besides the above-mentioned fact that each such Hamiltonian admits many non-equivalent physical interpretations realized via mutually nonequivalent Hilbert spaces $\mathcal{H}^{(P)}$, one can also start from a *fixed* self-adjoint Hamiltonian \mathfrak{h} defined in $\mathcal{H}^{(P)}$ and move towards *many* alternative isospectral images defined, in some (different) Hilbert space \mathcal{H} , by formula $\hat{H} = \Omega^{-1} \mathfrak{h} \Omega$. Here the so called Dyson map Ω should be assumed nontrivial i.e., non-unitary: $\Omega^\dagger \Omega = \Theta \neq \mathbb{1}$.

In phenomenology and practice, the only reason for preference and choice between \hat{H} and \mathfrak{h} is the feasibility of calculations and the constructive nature of experimental predictions. However, we also should be aware of the fact that Ω is rather often an unbounded operator, so that many mathematical subtle points usually arise when moving from \mathfrak{h} to \hat{H} in the way suggested above because of, among others, *domain details*. Examples of this kind of problems are discussed in [13, 14, 15] in connection with the so-called *pseudo-bosons*.

2.1. The three-space scenario

For certain complicated quantized systems (say, of tens or hundreds of fermions as occur, typically, in nuclear physics, or for many-body systems) the traditional theory forces us to work with an almost prohibitively complicated Hilbert space $\mathcal{H}^{(P)}$ which is not “friendly” at all. Typically, this space acquires the form of a multiple product $\bigotimes L^2(\mathbb{R}^3)$ or, even worse, of an antisymmetrized Fock space. In such a case, unfortunately, wave functions $\psi^{(P)}(t)$ in $\mathcal{H}^{(P)}$ become hardly accessible to explicit construction. The technical difficulties make Schrödinger’s equation practically useless: no time evolution can be easily deduced.

A sophisticated way towards a constructive analysis of similar quantum systems has been described by Scholtz et al [11]. They felt inspired by the encouraging practical experience with the so called Dyson’s mappings $\Omega^{(Dyson)}$ between bosons and fermions in nuclear physics. Still, their technique of an efficient simplification of the theory is independent of any particular implementation details. One only has to assume that the *overcomplicated* realistic Hilbert space $\mathcal{H}^{(P)}$ is being mapped

on a *much simpler*, friendly intermediate space $\mathcal{H}^{(F)}$. The latter space remains just auxiliary and unphysical but it renders the calculations (e.g., of spectra) feasible. In particular, the complicated state vectors $\psi^{(P)}(t)$ are made friendlier via an invertible transition from $\mathcal{H}^{(P)}$ to $\mathcal{H}^{(F)}$,

$$\psi^{(P)}(t) = \Omega \psi(t) \in \mathcal{H}^{(P)}, \quad \psi(t) \in \mathcal{H} = \mathcal{H}^{(F)}. \quad (2.1)$$

The introduction of the redundant superscript $^{(F)}$ underlines the maximal friendliness of the space (note, e.g., that in the above-mentioned nuclear-physics context of Ref. [11], the auxiliary Hilbert space $\mathcal{H}^{(F)}$ was a bosonic space). It is clear that, since Ω is not unitary, the inner product between two functions $\psi_1^{(P)}(t)$ and $\psi_2^{(P)}(t)$ (treated as elements of $\mathcal{H}^{(P)}$) differs from the one between $\psi_1(t)$ and $\psi_2(t)$,

$$\langle \psi_1^{(P)}, \psi_2^{(P)} \rangle_P = \langle \psi_1, \Omega^\dagger \Omega \psi_2 \rangle_F \neq \langle \psi_1, \psi_2 \rangle_F. \quad (2.2)$$

Here we use the suffixes P and F to stress the fact that the Hilbert spaces $\mathcal{H}^{(F)}$ and $\mathcal{H}^{(P)}$ are different and, consequently, they have different inner products in general. Under the assumption that $\ker\{\Omega\}$ only contains the zero vector, and assuming for the moment that Ω is bounded, $\Theta := \Omega^\dagger \Omega$ may be interpreted as producing another, alternative inner product between elements $\psi_1(t)$ and $\psi_2(t)$ of $\mathcal{H}^{(F)}$. This is because Θ is strictly positive. This suggests to introduce a third Hilbert space, $\mathcal{H}^{(S)}$, which coincides with $\mathcal{H}^{(F)}$ but for the inner product. The new product, $\langle \cdot, \cdot \rangle^{(S)}$, is defined by formula

$$\langle \psi_1, \psi_2 \rangle^{(S)} := \langle \psi_1, \Theta \psi_2 \rangle_F \quad (2.3)$$

and, because of (2.2), exhibits the following unitary-equivalence property

$$\langle \psi_1, \psi_2 \rangle^{(S)} \equiv \langle \psi_1^{(P)}, \psi_2^{(P)} \rangle_P. \quad (2.4)$$

More details on this point can be found in [9]. It is worth stressing that the fact that Ω is bounded makes it possible to have Θ everywhere defined in $\mathcal{H}^{(F)}$. Under the more general (and, we should say, more common) situation when Ω is not bounded, we should be careful about the possibility of introducing Θ , since $\Omega^\dagger \Omega$ could not be well defined [16]: indeed, for some f in the domain of Ω , $f \in D(\Omega)$, we could have that $\Omega f \notin D(\Omega^\dagger)$. If this is the case, the best we can have is that the inner product $\langle \cdot, \cdot \rangle^{(S)}$ is defined on a dense subspace of $\mathcal{H}^{(F)}$.

2.2. A redefinition of the conjugation

The adjoint X^\dagger of a given (bounded) operator X acting on a certain Hilbert space \mathcal{K} , with inner product (\cdot, \cdot) , is defined by the following equality:

$$(X\varphi, \Psi) = (\varphi, X^\dagger\Psi).$$

Here φ and Ψ are arbitrary vectors in \mathcal{K} . It is clear that changing the inner product also produces a different adjoint. Hence the adjoint in $\mathcal{H}^{(S)}$ is different from that in $\mathcal{H}^{(F)}$, since their inner products are different. The technical simplifying assumption that X is bounded is ensured, for instance, if we consider finite dimensional Hilbert spaces. In this way we can avoid difficulties which could arise, e.g., due to the

unboundedness of the metric operator Θ . The choice of $\dim \mathcal{H}^{(P,F,S)} < \infty$ gives also the chance of getting analytical results which, otherwise, would be out of our reach.

Once we have, in $\mathcal{H}^{(P)}$, the physical, i.e., safely hermitian and self-adjoint $\mathfrak{h} = \Omega \hat{H} \Omega^{-1} = \mathfrak{h}^\dagger$, we may easily deduce that

$$\hat{H} = \Theta^{-1} \hat{H}^\dagger \Theta := \hat{H}^\ddagger \quad (2.5)$$

where † stands for the conjugation in either $\mathcal{H}^{(P)}$ or $\mathcal{H}^{(F)}$ while ‡ may be treated as meaning the conjugation in $\mathcal{H}^{(S)}$ which is metric-mediated (sometimes also called “non-Dirac conjugation” in physics literature). Any operator \hat{H} which satisfies Eq. (2.5) is said to be *quasi-Hermitian*. The similarity of superscripts † and ‡ emphasizes the formal parallels between the three Hilbert spaces $\mathcal{H}^{(P,F,S)}$.

Once we temporarily return to the point of view of physics we must emphasize that the use of the nontrivial metric Θ is strongly motivated by the contrast between the simplicity of \hat{H} (acting in friendly $\mathcal{H}^{(F)}$ as well as in the non-equivalent but physical $\mathcal{H}^{(S)}$) and the practical intractability of its isospectral partner \mathfrak{h} (defined as acting in a constructively inaccessible Dyson-image space $\mathcal{H}^{(P)}$). Naturally, once we make the selection of $\mathcal{H}^{(S)}$ (in the role of the space in which the quantum system in question is represented), all of the other operators of observables (say, $\hat{\Lambda}$) acting in $\mathcal{H}^{(S)}$ must be also self-adjoint in the same space, i.e., they must be quasi-Hermitian with respect to *the same* metric,

$$\hat{\Lambda} = \hat{\Lambda}^\ddagger := \Theta^{-1} \hat{\Lambda}^\dagger \Theta. \quad (2.6)$$

In opposite direction, once we start from a given Hamiltonian \hat{H} and search for a metric Θ which would make it quasi-Hermitian (i.e., compatible with the requirement (2.5)), we reveal that there exist *many different* eligible metrics $\Theta = \Theta(\hat{H})$. In such a situation the simultaneous requirement of the quasi-Hermiticity of another operator imposes new constraints of the form $\Theta(\hat{H}) = \Theta(\hat{\Lambda})$ which restricts, in principle, the ambiguity of the metric [11]. Thus, a finite series of the quasi-Hermiticity constraints

$$\hat{\Lambda}_j = \hat{\Lambda}_j^\ddagger, \quad j = 1, 2, \dots, J \quad (2.7)$$

often leads to a unique physical metric Θ and to the unique, optimal Hilbert-space representation $\mathcal{H}^{(S)}$. This is what naturally extends the usual requirement of the ordinary textbook quantum mechanics which requires the set of the observables to be self-adjoint in a *pre-selected* Hilbert-space representation or very concrete realization $\mathcal{H}^{(F)}$.

Let us now briefly describe some consequences of (2.5) to the dynamical analysis of \mathcal{S} . As we have already seen relation (2.3) defines the *physical* inner product. We will now verify that this is the natural inner product to be used to find the expected unitary evolution generated by the quasi-Hermitian operator \hat{H} .

The first consequence of property $\hat{H} = \Theta^{-1} \hat{H}^\dagger \Theta$ is that $e^{i\hat{H}^\dagger t} = \Theta e^{i\hat{H} t} \Theta^{-1}$. The proof of this equality is trivial whenever the operators involved are bounded, condition which we will assume here for simplicity (as stated above, we could

simply imagine that our Hilbert spaces are finite-dimensional). Condition (2.5) allows us to keep most of the standard approach to the dynamics of the quantum system sketched in the Introduction, even in presence of a manifest non-Hermiticity of \hat{H} in friendly $\mathcal{H}^{(F)}$. Indeed, once we consider the Schrödinger evolution of a vector $\Psi(0)$, which we take to be $\Psi(t) = e^{-i\hat{H}t}\Psi(0)$, we may immediately turn attention to the time-dependence of the *physical* norm in $\mathcal{H}^{(S)}$,

$$\begin{aligned}\|\Psi(t)\|^2 &:= \langle \Psi(t), \Psi(t) \rangle^{(S)} = \langle e^{-i\hat{H}t}\Psi(0), \Theta e^{-i\hat{H}t}\Psi(0) \rangle_F = \\ &= \langle \Psi(0), e^{i\hat{H}^\dagger t} \Theta e^{-i\hat{H}t}\Psi(0) \rangle_F = \langle \Psi(0), \Theta \Psi(0) \rangle_F = \|\Psi(0)\|^2,\end{aligned}$$

Thus, for all $\Psi \in \mathcal{H}^{(S)}$ the natural use of the inner product $\langle \cdot, \cdot \rangle^{(S)}$ and of the related norm gives a probability which is preserved in time. This observation has also an interesting (even if expected) consequence: the dual evolution, i.e. the time evolution of the observables in the Heisenberg representation, has the standard form, $X(t) = e^{i\hat{H}t} X e^{-i\hat{H}t}$, so that $\dot{X}(t) = i[\hat{H}, X(t)]$. This is true independently of the fact that \hat{H} is self-adjoint or not, as in the present case. Indeed if we ask the mean value (in the product (S)) of the observables to be independent of the representation adopted, i.e. if we require that

$$\langle \varphi(t), X \Psi(t) \rangle^{(S)} = \langle e^{-i\hat{H}t} \varphi(0), \Theta X e^{-i\hat{H}t} \Psi(0) \rangle = \langle \varphi(0), X(t) \Psi(0) \rangle^{(S)}$$

then we are forced to put $X(t) = e^{i\hat{H}t} X e^{-i\hat{H}t}$ (rather than the maybe more natural $e^{i\hat{H}t} X e^{-i\hat{H}^\dagger t}$). This means that a consistent approach to the dynamical problem can be settled up also when the energy operator of the system is not self-adjoint, paying the price by replacing the Hilbert space in which the model was first defined, and its inner product, with something slightly different. In this different Hilbert space the time evolution *does its job*, preserving probabilities and satisfying the usual differential equations, both in the Schrödinger and in the Heisenberg pictures.

2.3. PT-symmetric quantum mechanics

One of the most efficient suppressions of the ambiguity of the metric has been proposed within the PTSQM formalism where the role of the additional observable $\hat{\Lambda}$ is being assigned to another involution \mathbf{C} [7]. The presence as well as a “hidden” mathematical meaning of the original involution \mathbf{P} may be further clarified by introducing, together with our three Hilbert spaces $\mathcal{H}^{(P,F,S)}$, also another auxiliary space \mathbf{K} with the structure of the Krein space endowed with an invertible indefinite metric equal to the parity operator as mentioned above, $\mathbf{P} = \mathbf{P}^\dagger$ [17]. One requires that the given Hamiltonian proves \mathbf{P} -self-adjoint in \mathbf{K} , i.e., that it satisfies equation

$$\hat{H}^\dagger \mathbf{P} = \mathbf{P} \hat{H}. \quad (2.8)$$

This is an intertwining relation between two non self-adjoint operators \hat{H} and \hat{H}^\dagger , and \mathbf{P} is the *intertwining operator* (IO, [18]). In the standard literature on IO, see [19] for instance, the operators intertwined by the IO are self-adjoint, so that their eigenvalues are real, coincident, and the associated eigenvectors are orthogonal (if the degeneracy of each eigenvalue is 1). Here, see below, these eigenvalues are not

necessarily real. Nevertheless, many situations do exist in the literature in which the eigenvalues of some non self-adjoint operator can be computed and turn out to be real, [13, 14, 15, 20], even if \hat{H} is not self-adjoint in $\mathcal{H}^{(F)}$.

Let us go back to the requirement (2.8). Multiple examples of its usefulness may be found scattered in the literature [20]. Buslaev and Grecchi [21] were probably the first mathematical physicists who started calling property (2.8) of the Hamiltonian a “**PT**–symmetry”. Let us now briefly explain its mathematical benefits under a not too essential additional assumption that the spectrum $\{E_n\}$ of \hat{H} is discrete and non-degenerate (though, naturally, not necessarily real). Then we may solve the usual Schrödinger equation for the (right) eigenvectors of \hat{H} ,

$$\hat{H} \psi_n = E_n \psi_n, \quad n = 0, 1, \dots \quad (2.9)$$

as well as its Schrödinger-like conjugate rearrangement

$$\hat{H}^\dagger \psi^n = E_n^* \psi^n, \quad n = 0, 1, \dots \quad (2.10)$$

In the light of Krein-space rule (2.8) the latter equation acquires the equivalent form

$$\hat{H} (\mathbf{P}^{-1} \psi^n) = E_n^* (\mathbf{P}^{-1} \psi^n), \quad n = 0, 1, \dots \quad (2.11)$$

so that we may conclude that

- either $E_n = E_n^*$ is real and the action of \mathbf{P}^{-1} on ψ^n gives merely a vector proportional to the ψ_n ,
- or $E_n \neq E_n^*$ is not real. In this regime we have $E_n^* = E_m$ at some $m \neq n$. This means that the action of \mathbf{P}^{-1} on the left eigenket ψ^n produces a right eigenvector of \hat{H} which is proportional to certain right eigenket ψ_m of Eq. (2.9) at a *different* energy, $m \neq n$.

We arrived at a dichotomy: Once we take all of the n –superscripted left eigenvectors ψ^n of \hat{H} and premultiply them by the inverse pseudometric, we obtain a new set of ket vectors $\phi_n = \mathbf{P}^{-1} \psi^n$ which are either all proportional to their respective n –subscripted partners ψ_n (while the whole spectrum is real) or not. This is a way to distinguish between two classes of Hamiltonians. Within the framework of quantum mechanics only those with the former property of having real eigenvalues may be considered physical (see [14] for a typical illustration). In parallel, examples with the latter property may be still found interesting beyond the limits of quantum mechanics, i.e., typically, in classical optics [22]. The recent growth of interest in the latter models (exhibiting the so called spontaneously broken **PT**–symmetry) may be well documented by their presentation via the dedicated webpages [23].

The Hamiltonians \hat{H} with the real and non-degenerate spectra may be declared acceptable in quantum mechanics. In the subsequent step our knowledge of the eigenvectors ψ^n of \hat{H} enables us to define the positive-definite operator of the metric directly [24],

$$\Theta = \sum_{n=0}^{\infty} |\psi^n\rangle \langle \psi^n|. \quad (2.12)$$

As long as this formula defines different matrices for different normalizations of vectors $|\psi^n\rangle$ (cf. [25] for details) we may finally eliminate this ambiguity via the factorization ansatz

$$\Theta = \mathbf{P}\mathbf{C} \quad (2.13)$$

followed by the *double* involutivity constraint [7]

$$\mathbf{P}^2 = I, \quad \mathbf{C}^2 = I. \quad (2.14)$$

We should mention that the series in (2.12) could be just formal. This happens whenever the operator Θ is not bounded. This aspect was considered in many details in connection with pseudo-bosons, see [13] for instance, where one of us has proved that Θ being bounded is equivalent to $\{\Psi^n\}$ being a Riesz basis.

Skipping further technical details we may now summarize: The operators of the form (2.13) + (2.14) have to satisfy a number of additional mathematical conditions before they may be declared the admissible metric operators determining the inner product in $\mathcal{H}^{(S)}$. *Vice versa*, once we satisfy these conditions (cf., e.g., Refs. [11, 26] for their list) we may replace the unphysical and auxiliary Hilbert space $\mathcal{H}^{(F)}$ and the intermediate Krein space \mathbf{K} (with its indefinite metric $\mathbf{P} = \mathbf{P}^\dagger$) by the ultimate physical Hilbert space $\mathcal{H}^{(S)}$ of the PTSQM theory. The input Hamiltonian \hat{H} may be declared self-adjoint in $\mathcal{H}^{(S)}$. In this space it also generates the correct unitary time-evolution of the system in question, at least for the time-independent interactions, [27].

3. Summary

We have given here a brief review of some aspects of PTSQM with particular interest to the role of different inner products and their related conjugations. We have also discussed how the time evolution of a system with a non self-adjoint Hamiltonian can be analyzed within this settings, and the role of the different inner products is discussed.

Among other lines of research, we believe that the construction of algebras of unbounded operators associated to PTSQM, along the same lines as in [3], is an interesting task and we hope to be able to do that in the near future.

Acknowledgment

Work supported in part by the GAČR grant Nr. P203/11/1433, by the MŠMT “Doppler Institute” project Nr. LC06002 and by the Inst. Res. Plan AV0Z10480505 and in part by M.I.U.R.

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